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# EXACT SOLUTION OF THE PROBLEM ON A SIX-CONSTANT JEFFREYS MODEL OF FLUID IN A PLANE CHANNEL

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*An exact analytic solution of the problem of a generalized viscoelastic Jeffreys fluid flow in a plane channel under the action of a pressure gradient is found. The velocity profiles are obtained in a parametric form with a velocity gradient taken as a parameter. The critical values of the pressure gradient are determined, which, when exceeded, lead to weak tangential discontinuities in the longitudinal velocity profile. When the pressure gradient changes smoothly over some range of parameters, a hysteresis loop emerges on the graph of the flow rate versus the pressure gradient.*

**Introduction.** The problems of modeling and stability of viscoelastic fluids are of great importance in rheology and processing technology of polymeric materials. Special attention is given to the so-called spurt effects, i.e., emerging instabilities of fluid outflow from capillaries and channels, and their reasons [1–5]. Mathematical modeling of these effects requires the investigation in the existence, uniqueness, and stability of the solutions of equations of motion of viscoelastic fluids.

The present paper studies the flow of a generalized viscoelastic Jeffreys fluid with an objective time derivative $F_{abc}$ ($a$, $b$, and $c$ are arbitrary material constants). This version of the rheological model is a particular case of the eight-constant Oldroyd model, where the time derivatives of extrastresses and strain rates comprise the same material constants. Using the notation introduced in [6], the transition from the eight-constant Oldroyd model to the six-constant generalized Jeffreys model occurs under the following conditions:

$$\mu_1 = -a\lambda_1, \quad \mu_0 = c\lambda_1, \quad \nu_1 = b\lambda_1, \quad \mu_2 = -a\lambda_2, \quad \nu_2 = b\lambda_2.$$

In turn, Maxwell, De Witt, and White–Metzner models, as well as three- and four-constant Oldroyd models can be treated as particular cases of the generalized six-constant Jeffreys model.

Our aim was to obtain an analytical solution of equations of motion for the six-constant Jeffreys model on the assumption of a one-dimensional flow in a plane channel and to select kinematically admissible and physically realizable solutions from the set of solutions obtained by applying well-known results of the stability theory. In particular, it was shown in [7–9] that, for some viscoelastic fluids, there exist kinematically admissible but unstable and not naturally realizable motions. From the analysis of the four-constant Oldroyd model simulating fluid in a round pipe [10], there follows the ambiguity of the velocity-gradient distribution in the radial direction. The obtained profiles of the longitudinal velocity either are smooth and almost parabolic or display weak discontinuities in the tangential direction, depending on the magnitude of the pressure gradient and the ratio of the delay time to the relaxation time. From the numerical calculations of [10], it also follows that the curve of the flow rate versus the pressure gradient has a hysteresis loop.

Studying a viscoelastic Maxwell fluid flow past a body, Ultmann and Denn [11] showed that the second elasticity number $El_2$ is critical. In a subcritical case ($El_2 < 1$), equations admit continuous solutions. In the postcritical case ($El_2 > 1$), equations are hyperbolic, and their solutions are characterized by significant tangential discontinuities. Apparently, similar critical regimes may also exist in viscoelastic fluid flows in pipes and channels.

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**1. Formulation of the Problem and Its Exact Solution.** Let us consider the exact formulation of the problem of a viscoelastic fluid flow in a plane channel under the action of a pressure gradient. The equations of motion in terms of extrastresses have the form [12]

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \nabla \cdot T,$$

where $\rho$ is the density, $\boldsymbol{v}$ is the velocity vector, $p$ is the mechanical pressure, and $T$ is the tensor of extrastresses.

The fluid is assumed to be incompressible:

$$\nabla \cdot \boldsymbol{v} = 0.$$

The Jeffreys equation with the most general associated time derivative retaining symmetry has the form $T + \lambda_1 F_{abc} T = \mu(D + \lambda_2 F_{abc} D)$ or is written in an expanded form

$$T + \lambda_1 \Big[\frac{dT}{dt} - W \cdot T + T \cdot W + a(T \cdot D + D \cdot T) + bT : D\boldsymbol{I} + cD \operatorname{tr} T\Big]$$

$$= 2\mu\Big[D + \lambda_2\Big(\frac{dD}{dt} - W \cdot D + D \cdot W + 2aD \cdot D + bD : D\boldsymbol{I}\Big)\Big],$$

where $\nabla\boldsymbol{v} = D + W$ is the velocity gradient, $D = (\nabla\boldsymbol{v}^{\rm t} + \nabla\boldsymbol{v})/2$ is the symmetric part of $\nabla\boldsymbol{v}$, $W = (\nabla\boldsymbol{v}^{\rm t} - \nabla\boldsymbol{v})/2$ is the antisymmetric part $\nabla\boldsymbol{v}$, $\mu$ is the viscosity, $\lambda_1$ is the relaxation time, and $\lambda_2$ is the delay time.

In Cartesian coordinates, a unidirectional [$v_z(x)$ and $v_x = v_y = 0$] steady-state flow in a long plane channel with dimensions $2h \times \omega \times L$ is described by the equation

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = \frac{\partial p}{\partial z}. \qquad (1)$$

In the case of constant viscosity $\mu$, the components of the extrastress tensor do not depend on $z$, and all the derivatives with respect to $z$, except for $dp/dz = -R = \text{const}$, are zero. Equation (1) may be integrated once:

$$\tau_{xz} = -Rx. \qquad (2)$$

The extrastress-tensor components are obtained as follows:

$$\tau_{xx} + \tau_{xz}\lambda_1 \frac{dv_z}{dx}(1 + a + b) = \mu\lambda_2\Big(\frac{dv_z}{dx}\Big)^2(1 + a + b), \quad \tau_{yy} + \tau_{xz}\lambda_1 \frac{dv_z}{dx}\, b = \mu\lambda_2\Big(\frac{dv_z}{dx}\Big)^2 b,$$

$$\tau_{zz} + \tau_{xz}\lambda_1 \frac{dv_z}{dx}(-1 + a + b) = \mu\lambda_2\Big(\frac{dv_z}{dx}\Big)^2(-1 + a + b); \qquad (3)$$

$$\tau_{xz} + \frac{1}{2}\lambda_1 \frac{dv_z}{dx}[\tau_{zz} - \tau_{xx} + (\tau_{zz} + \tau_{xx})(a + c)] = \mu \frac{dv_z}{dx}. \qquad (4)$$

By using (3) and substituting (4) into (2), we find

$$\tau_{xz} = \mu \frac{dv_z}{dx}\, \frac{1 + \lambda_1\lambda_2[1 - (a + b)(a + c)](dv_z/dx)^2}{1 + \lambda_1^2[1 - (a + b)(a + c)](dv_z/dx)^2} = -Rx. \qquad (5)$$

From (3)–(5), the expressions for the effective viscosity and the first and second differences of the normal stress can be obtained:

$$\eta \frac{dv_z}{dx} = \mu \frac{1 + \lambda_1\lambda_2[1 - (a + b)(a + c)](dv_z/dx)^2}{1 + \lambda_1^2[1 - (a + b)(a + c)](dv_z/dx)^2},$$

$$\Psi_1 = \frac{\tau_{zz} - \tau_{xx}}{(dv_z/dx)^2} = \frac{2\mu(\lambda_1 - \lambda_2)}{1 + \lambda_1^2[1 - (a + b)(a + c)](dv_z/dx)^2}, \qquad (6)$$

$$\Psi_2 = \frac{\tau_{xx} - \tau_{yy}}{(dv_z/dx)^2} = \frac{-\mu(1 + a)(\lambda_1 - \lambda_2)}{1 + \lambda_1^2[1 - (a + b)(a + c)](dv_z/dx)^2}.$$

For a flow where $dv_z/dx$ is a specified quantity, Eqs. (6) are the problem solution. For a flow with a specified pressure gradient, the velocity gradient and profile should be found from the solution of the equation of motion (5). In this case, we assume that $(a + b)(a + c) \neq 1$. In the opposite case, we have a trivial solution with a parabolic velocity profile.

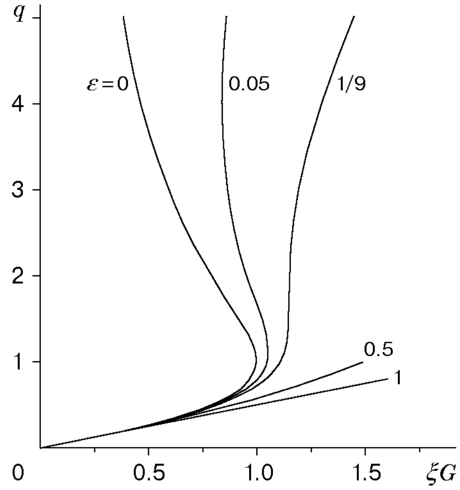


Fig. 1. Shear velocity versus pressure gradient [formula (7)] for various values of $\varepsilon$.

Introducing the dimensionless parameters $\xi = x/h$, $\varepsilon = \lambda_2/\lambda_1$, $A = \sqrt{1-(a+b)(a+c)}$, and dimensionless velocity $w = v_z\lambda_1 A/h$, shear velocity $q = \lambda_1 A dv_z/dx$, and pressure gradient $G = 2R\lambda_1 Ah/\mu$, we obtain

$$2q(1+\varepsilon q^2)/(1+q^2) = -G\xi. \tag{7}$$

This relation (with accuracy to the notation) was first obtained for the four-constant Oldroyd model in [8]. For the given pressure differential, it describes the distribution of the dimensionless velocity gradient as a function of the dimensionless coordinate $\xi$. Equation (7) is nonlinear and can have several solutions.

For $\varepsilon < 1/9$, the function $q = q(G\xi)$ is defined in the range between the maximum and the minimum values of $G\xi$, where the dimensionless velocity gradient takes three values (Fig. 1). The lower the value of $\varepsilon$, the wider the ambiguity zone and the greater the velocity gradient jump. This implies that the velocity profiles must have discontinuities or loops. For $\varepsilon = 0$ and $0 < A < 1$, there are either two solutions ($G < 1$), or there are no stationary solutions ($G > 1$) at all.

It is possible to calculate from (7) the derivative

$$\frac{d\xi}{dq} = -\frac{2}{G}\,\frac{1-q^2+3\varepsilon q^2+\varepsilon q^4}{(1+q^2)^2},$$

and by integrating the equation

$$w = \int q\,d\xi + C = \int q\,\frac{d\xi}{dq}\,dq + C,$$

we can find the longitudinal velocity as a function of the velocity gradient:

$$w = \{\varepsilon(1+q^2) + (\varepsilon-1)[\ln|1+q^2| + 2/(1+q^2)] + C\}/G.$$

The integration constant $C$ is obtained from the boundary condition $w = 0$ and $q = q_{\mathrm{w}}$ at $\xi = \pm 1$:

$$C = -\varepsilon(1+q_{\mathrm{w}}^2) - (\varepsilon-1)[\ln|1+q_{\mathrm{w}}^2| + 2/(1+q_{\mathrm{w}}^2)].$$

At the same time, from Eq. (7), the transverse coordinate $\xi$ can be determined as a function of $q$:

$$\xi = -\frac{2q(1+\varepsilon q^2)}{1+q^2}\,\frac{1}{G}.$$

The resultant representation of the velocity profile may be in a parametric form

$$w = \frac{1+q_{\mathrm{w}}^2}{2q_{\mathrm{w}}(1+\varepsilon q_{\mathrm{w}}^2)}\left[\varepsilon(q_{\mathrm{w}}^2-q^2) + (1-\varepsilon)\left(\ln\left|\frac{1+q^2}{1+q_{\mathrm{w}}^2}\right| + \frac{2(q_{\mathrm{w}}^2-q^2)}{(1+q^2)(1+q_{\mathrm{w}}^2)}\right)\right], \quad \xi = \frac{q(1+\varepsilon q^2)(1+q_{\mathrm{w}}^2)}{q_{\mathrm{w}}(1+q^2)(1+\varepsilon q_{\mathrm{w}}^2)}, \tag{8}$$

where the dimensionless pressure gradient $G$ and the velocity gradient on the wall $q_{\mathrm{w}}$ are related as

$$2q_{\mathrm{w}}(1+\varepsilon q_{\mathrm{w}}^2)/(1+q_{\mathrm{w}}^2) = -G. \tag{9}$$

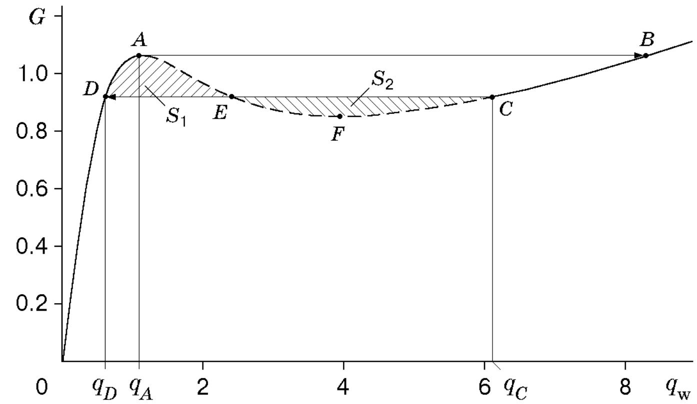


Fig. 2. Pressure gradient $G$ versus shear velocity on the channel wall $q_{\rm w}$ for $\varepsilon = 0.05$.

For $\varepsilon < 1/9$, one or three values of the velocity gradient on the wall $q_{\rm w}$ may correspond to the specified value of the pressure gradient $G$.

**2. Solution Analysis.** Thus, an analytical solution has been obtained in a parametric form, which describes a set of longitudinal velocity profiles including those with tangential discontinuities.

As an example, we consider the dependence of the dimensionless pressure gradient $G$ on the parameter $q_{\rm w}$, plotted on the basis of Eq. (9) for $\varepsilon = 0.05$ (Fig. 2). On the descending part of the curve $G(q_{\rm w})$ between the points $A$ and $F$, and on the ascending segment of the curve between the points $F$ and $C$, the velocity profiles are kinematically inadmissible. An example of such profiles satisfying the equations of motion but exceeding channel boundaries is the velocity profile at the point $E$ (curve 3 in Fig. 4).

Therefore, motions can be realized corresponding to the ascending parts of the curve $G(q_{\rm w})$ from 0 to the point $A$ and from the point $C$ and higher. In Fig. 2, the segments corresponding to kinematically admissible linearly stable regimes are shown by the solid curve, and the segments corresponding to kinematically inadmissible regimes are denoted by the dashed curve.

We assume that the flow occurs at a specified pressure gradient. A smooth increase in the pressure gradient $G$ is accompanied by a subcritical bifurcation: the solution nonuniqueness appears before the critical value of the pressure gradient $G_{\max}$ is attained. Since branching off subcritical solutions are unstable for infinitesimal perturbation amplitudes, actually no continuous bifurcation is observed [13–15]. It can be assumed that perturbed solutions leaving the domain of solution attraction corresponding to the main flow will "skip" the unstable branched-off solution and tend to a stable solution with an average velocity significantly exceeding that in the main flow. Velocity profiles in this linearly stable regime inevitably have closed loops, which leads to weak tangential discontinuities (the function is continuous, and the derivative is discontinuous) or to strong tangential discontinuities (the function is discontinuous). The strong tangential discontinuities are unstable with respect to perturbations that are already infinitely small [13]. Therefore, only weak tangential discontinuities may occur, which correspond to the transition along the line $CD$ positioned so that the areas of the domains $S_1$ and $S_2$ are equal. Figure 3 shows the velocity profiles corresponding to the jump from the point $A$ to the point $B$, which were constructed using formulas (8) under the same pressure gradient $G_{\max}$ critical for this value of $\varepsilon$.

With a smooth decrease in the pressure gradient, the inverse transition to the left branch of the curve occurs at $G_{\rm eq} < G_{\max}$. At this pressure gradient, the branching point of the second solution proves to be on the channel boundary, and the velocities inside the channel fully correspond to the first solution (Fig. 4). (In Figs. 3 and 4, the dashed curves denote unstable and kinematically inadmissible velocity profiles.) Thus, with a smooth decrease in the pressure gradient, there is no velocity-profile reconstruction.

The dimensionless flow rate $Q = Q'\lambda_1 A/(h^2\omega)$ can be calculated from the obtained velocity profile by integrating the velocity profile over the channel depth. For active loading ($dG > 0$) at $G < G_{\max}$ and for the case of unloading ($dG < 0$) at $G < G_{\rm eq}$, the integrand is smooth for every $\varepsilon$, and the definite integral is calculated over the entire channel depth:

$$Q = 2\int_0^1 w\,d\xi = 2\int_0^{q_{\rm w}} w\,\frac{d\xi}{dq}\,dq. \qquad (10)$$

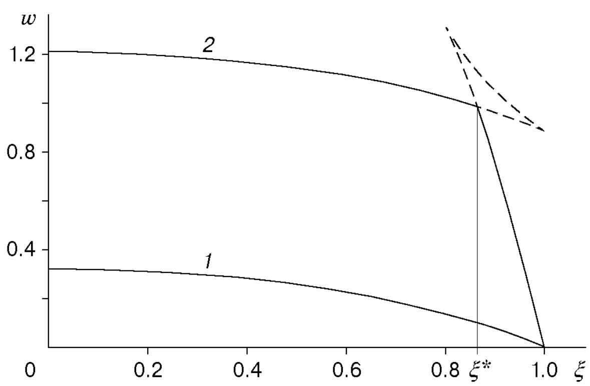


Fig. 3

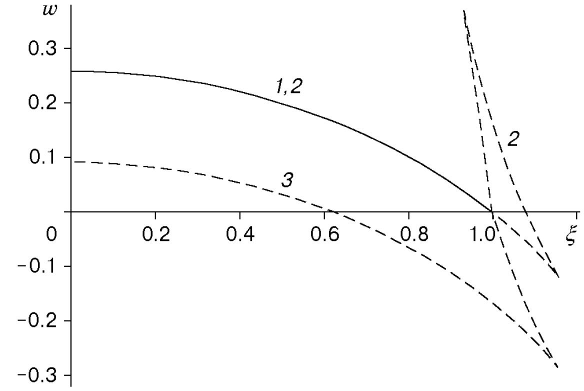


Fig. 4

Fig. 3. Velocity profiles ($G = G_{\max} = 1.0559$ and $\varepsilon = 0.05$) at the points $A$ (1) and $B$ (2).

Fig. 4. Velocity profiles ($G = G_{\rm eq} = 0.9111$ and $\varepsilon = 0.05$) at the points $D$ (1), $C$ (2), and $E$ (3).

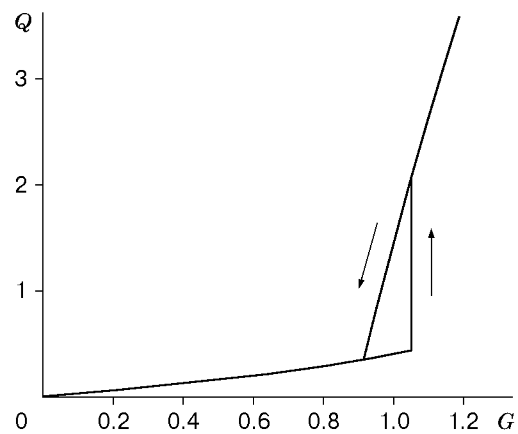


Fig. 5

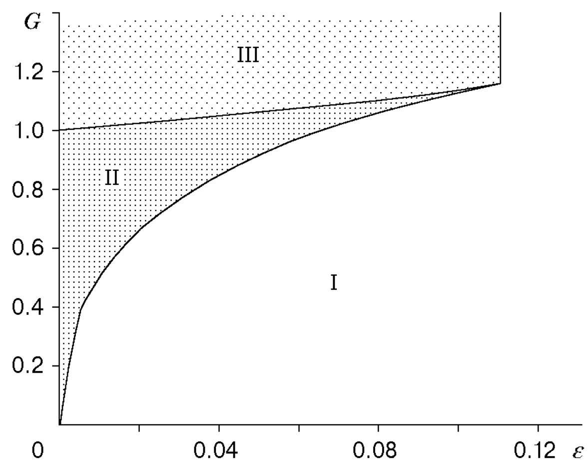


Fig. 6

Fig. 5. Hysteresis loop on the curve of flow rate versus dimensional pressure gradient for $\varepsilon = 0.05$.

Fig. 6. Phase plane $G$–$\varepsilon$: I and III are the domains of stability and II is the domain of metastability.

In the case of active loading at $G > G_{\max}$ and also under unloading at $G > G_{\rm eq}$, the integrand for $\varepsilon < 1/9$ has a closed loop with a branching point inside the channel. In order to calculate the flow rate, the definite integral should be broken into a sum of two integrals with the limits from 0 to $q_D$ and from $q_C$ to $q_{\rm w}$, where the integrand is monotonic:

$$Q = 2\int\limits_0^{\xi^*} w\,d\xi + 2\int\limits_{\xi^*}^{1} w\,d\xi = 2\int\limits_0^{q_D} w\,\frac{d\xi}{dq}\,dq + 2\int\limits_{q_C}^{q_{\rm w}} w\,\frac{d\xi}{dq}\,dq. \tag{11}$$

Here $\xi^*$ is the dimensionless coordinate of the branching point; $q_D$ and $q_C$ are the values of the parameter $q$ corresponding to it.

The dependence of the flow rate on the dimensionless pressure gradient is also determined in a parametric form from relations (10) or (11) with allowance for (9).

Since the transitions from one branch of the curve $Q(G)$ of the flow rate versus the dimensional pressure gradient to the other with increasing or decreasing pressure gradient occur at different critical values of $G$, a hysteresis loop appears on the curve $Q(G)$ (Fig. 5).

For $\varepsilon < 1/9$, there exists a certain interval $G_{\rm eq}$–$G_{\max}$, where the mainstream motion is metastable with respect to infinitisemal perturbations and unstable with respect to finite-amplitude perturbations. For this flow type, critical is the empiric condition $G^* = 1 + 0.2376\lambda_2/\lambda_1$. In particular, the critical condition for the De Witt model is $G^* = 1$. In the subcritical regime ($G \leqslant G^*$), the velocity profiles are continuous, smooth, and almost parabolic. In the postcritical regime ($G > G^*$), at $(a + b)(a + c) < 1$, the velocity profiles in the general case have weak tangential discontinuities, and there are no kinematically admissible solutions for Maxwell-type models ($\lambda_2 = 0$).

In the phase plane $G$–$\varepsilon$, one can identify three domains (Fig. 6): in domain I, there is a unique solution, the velocity profile is smooth and almost parabolic; in domain II, there are two linearly stable solutions, which are unstable with respect to finite-amplitude perturbations; in domain III, there is a stable unique solution, and there are tangential discontinuities in the velocity profile.

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